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The Open Descendants of Non-Diagonal $SU(2)$ WZW Models

G. Pradisi[a,b,c] , A. Sagnotti[a,b] and Ya.S. Stanev[a,b,1]

[a] *Dipartimento di Fisica*
Università di Roma "Tor Vergata"
I.N.F.N. - Sezione di Roma "Tor Vergata"
Via della Ricerca Scientifica, 1 00133 Roma ITALY

[b] *Centre de Physique Théorique*
Ecole Polytechnique
91128 Palaiseau FRANCE

[c] *Centro "Vito Volterra", Università di Roma "Tor Vergata"*

Abstract

We extend the construction of open descendants to the $SU(2)$ WZW models with non-diagonal left-right pairing, namely E_7 and the D_{odd} series in the ADE classification of Cappelli, Itzykson and Zuber. The structure of the resulting models is determined to a large extent by the "crosscap constraint", while their Chan-Paton charge sectors may be embedded in a general fashion into those of the corresponding diagonal models.

[1] I.N.F.N. Fellow, on Leave from Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, BG-1784 Sofia, BULGARIA.

Introduction

We have recently shown [1] how "open descendants" [2] can be associated to all diagonal $SU(2)$ WZW models [3]. In this letter we extend the program of ref. [4] to the remaining, non-diagonal, ones, thus completing the classification for all the ADE modular invariants of Cappelli, Itzykson and Zuber [5]. This is particularly instructive, since the ansatz suggested by ref. [6] is tailored to the diagonal case. The extension of ref. [7] to the non-diagonal minimal models of the ADE classification [5] will be discussed elsewhere. We also sketch a general method to handle the constraints and exhibit the relationship between these models and the corresponding diagonal ones. The transverse-channel amplitudes are determined to a large extent by the crosscap constraint of ref. [8], and the resulting open descendants are somewhat surprising, since their Chan-Paton [9] charge spaces are typically far smaller than one would naively expect.

Let us begin by reviewing the general features of the construction of open descendants. The starting point is a left-right symmetric two-dimensional conformal field theory whose spectrum is described by a torus partition function

$$T \;=\; \sum_{a,b} \chi_a(\tau)\; N_{ab}\; \bar{\chi}_b(\bar{\tau}) \tag{1}$$

invariant under the modular group $PSL(2,Z)$. As is well known, this group is generated by the transformations

$$T:\; \tau \rightarrow \tau+1 \qquad \text{and} \qquad S:\; \tau \rightarrow -\frac{1}{\tau} \quad , \tag{2}$$

that act on the characters χ_a via two matrices, also denoted by T and S. In moving to the open descendants, one begins by projecting the closed spectrum with a direct-channel Klein-bottle amplitude K, that has the general form

$$K \;=\; \frac{1}{2}\sum_a \chi_a\; K_a \qquad \stackrel{S}{\longrightarrow} \qquad \tilde{K} \;=\; \frac{1}{2}\sum_a \chi_a\,(\Gamma_a)^2 \tag{3}$$

and, as indicated, is turned into the corresponding transverse-channel amplitude $\tilde{K}$ by an S transformation. In addition, the descendants contain "open" sectors, described by direct-channel annulus and Möbius amplitudes, that must respect the planar duality and

the factorization of disk amplitudes. The direct-channel annulus amplitude A has the general form

$$A \ = \ \frac{1}{2} \sum_{a,\beta,\gamma} \chi_a \ A_{a\beta\gamma} \ n_\beta \ n_\gamma \qquad \stackrel{S}{\longrightarrow} \qquad \tilde{A} \ = \ \frac{1}{2} \sum_a \chi_a \left(\sum_\beta \ B_{a\beta} \ n_\beta \right)^2 \tag{4}$$

and, as indicated, is turned into the transverse-channel amplitude $\tilde{A}$ by an S transformation. Finally, the direct-channel Möbius amplitude M has the general form

$$M \ = \ \pm \frac{1}{2} \sum_{a,\beta} \hat{\chi}_a \ M_{a\beta} \ n_\beta \qquad \stackrel{P}{\longrightarrow} \qquad \tilde{M} \ = \ \pm \frac{1}{2} \sum_a \hat{\chi}_a \ \Gamma_a \ \left(\sum_\beta \ B_{a\beta} \ n_\beta \right) \quad , \tag{5}$$

and is turned into the corresponding transverse-channel amplitude $\tilde{M}$ by the transformation $P : \frac{i\tau+1}{2} \rightarrow \frac{i+\tau}{2\tau}$, that acts on the characters via the matrix

$$P = T^{1/2} \ S \ T^2 \ S \ T^{1/2} \qquad . \tag{6}$$

The three matrices S, T and P satisfy the relations

$$S^2 \ = \ (ST)^3 \ = \ P^2 \ = \ C \qquad , \tag{7}$$

where C is the conjugation matrix. The resulting models extend 2D Conformal Field Theory to the open and unorientable case and are building blocks for the construction of open-string vacua. The three transverse-channel amplitudes $\tilde{K}$, $\tilde{A}$ and $\tilde{M}$ are closely related, since they describe the propagation of the closed spectrum along tubes terminating at holes and/or crosscaps. As a result, each coefficient in $\tilde{M}$ is the geometric mean of the corresponding ones in $\tilde{A}$ and in $\tilde{K}$. Moreover, the integer coefficients K_a, $M_{a\beta}$ and $A_{a\beta\gamma}$ describe consistently the (anti)symmetrization of the spectrum both in the closed and in the open sectors, provided they satisfy the relations

$$|K_a| \ = \ N_{aa} \qquad , \qquad M_{a\beta} \ = \ A_{a\beta\beta} \quad (\text{mod } 2) \qquad . \tag{8}$$

As discussed in ref. [1], this form of the last condition allows for multiple charge sectors corresponding to a single character. These constraints lead in general to rather complicated systems of diophantine equations for the tensors K_a, $M_{a\beta}$ and $A_{a\beta\gamma}$, whose solution determines the Chan-Paton charge space of the open sector, as well as the corresponding projection of the closed sector.

Descendants of the D_5 Model

Let us illustrate the construction on the simplest example, the D_5 model of level $k = 6$, whose torus partition function is

$$T = |\chi_1|^2 + |\chi_3|^2 + |\chi_5|^2 + |\chi_7|^2 + |\chi_4|^2 + (\chi_2\bar{\chi}_6 + h.c.) \tag{9}$$

where the subscript of χ_a is related to the isospin I by $a = 2I + 1$. This model admits two classes of open descendants corresponding to the two Klein-bottle projections

$$K^r = \frac{1}{2} (\chi_1 + \chi_3 + \chi_5 + \chi_7 - \chi_4) \tag{10}$$

and

$$K^c = \frac{1}{2} (\chi_1 + \chi_3 + \chi_5 + \chi_7 + \chi_4) \quad . \tag{11}$$

In both cases only the five self-conjugate characters χ_1, χ_3, χ_5, χ_7 and χ_4 are allowed in the transverse-channel annulus amplitude. Thus, in analogy with more elementary examples, both classes of models would be expected to contain five charge sectors. Rather surprisingly, however, they only admit four charge sectors. Indeed, turning on one charge at a time effectively splits the constraints into independent ones that link the *diagonal* elements in the annulus amplitude to the corresponding entries in the Möbius amplitude. Eqs. (3), (4), (5) and (8) may then be solved, and the consistent solutions may be combined into the general parametrization. The annulus and Möbius amplitudes are then

$$\begin{aligned} A^r = \frac{1}{2} \Big(& \chi_1(l_1^2 + l_2^2 + l_3^2 + l_4^2) + (\chi_2 + \chi_6)(2l_1l_2) + \\ & \chi_3(l_1^2 + 2l_1l_3 + 2l_1l_4 + 2l_3l_4) + \chi_4(2l_2l_3 + 2l_2l_4) + \\ & \chi_5(l_1^2 + l_3^2 + l_4^2 + 2l_1l_3 + 2l_1l_4) + \chi_7(l_1^2 + l_2^2 + 2l_3l_4) \Big) \quad , \end{aligned} \tag{12}$$

and

$$\begin{aligned} M^r = \pm \frac{1}{2} \Big(& \hat{\chi}_1(l_1 - l_2 + l_3 + l_4) + \hat{\chi}_3(-l_1) + \\ & \hat{\chi}_5(l_1 + l_3 + l_4) + \hat{\chi}_7(l_1 + l_2) \Big) \end{aligned} \tag{13}$$

for the model with all real charges, and

$$\begin{aligned} A^c \ &= \ \frac{1}{2}\Big(\chi_1(l_1^2+l_2^2+2l\bar{l}) \ + \ (\chi_2+\chi_6)(2l_1l_2) \ + \\ &\chi_3(l_1^2+l^2+\bar{l}^2+2l_1l+2l_1\bar{l}) \ + \ \chi_4(2l_2l+2l_2\bar{l}) \ + \\ &\chi_5(l_1^2+2l_1l+2l_1\bar{l}+2l\bar{l}) \ + \ \chi_7(l_1^2+l_2^2+l^2+\bar{l}^2)\Big) \end{aligned} \tag{14}$$

and

$$\begin{aligned} M^c \ &= \ \pm\frac{1}{2}\Big(\hat{\chi}_1(-l_1+l_2) \ + \ \hat{\chi}_3(l_1+l+\bar{l}) \ + \\ &\hat{\chi}_5(l_1) \ + \ \hat{\chi}_7(l_1+l_2+l+\bar{l})\Big) \end{aligned} \tag{15}$$

for the model with complex charges. It should be appreciated that in the latter model only the two real charges l_3 and l_4 turn into the complex pair $(l,\bar{l})$ where, as usual, $l=\bar{l}$. This is in sharp contrast with the behavior of the complex diagonal models, that contain at most one real charge. Although these amplitudes are not directly based on the fusion rules, an explicit study of the disk amplitudes shows that the open spectrum is nicely consistent with planar duality and factorization.

The Crosscap Constraint

In this Section we provide an alternative, rigorous derivation of the crosscap constraint of ref. [8]. Let us begin by noting that, in left-right symmetric models, all fields of the closed sector may be expressed in terms of chiral and antichiral vertex operators [10] according to

$$\phi_{\Delta,\bar{\Delta}}(z,\bar{z}) \ = \ \sum_{i,\bar{i},f,\bar{f}} {V_\Delta}^f{}_i(z)\ {\bar{V}_{\bar{\Delta}}}{}^{\bar{f}}{}_{\bar{i}}(\bar{z})\ \alpha_{i\bar{i}}{}^{f\bar{f}} \qquad , \tag{16}$$

where ${V_\Delta}^f{}_i$ denotes a chiral "field" of conformal dimension Δ acting on a state i and producing a state f and $\bar{V}$ denotes a corresponding antichiral "field". In order to analyze the behavior of the fields $\phi_{\Delta,\bar{\Delta}}(z,\bar{z})$ in the presence of a crosscap, it is convenient to introduce also

$$\phi_{\bar{\Delta},\Delta}(\bar{z},z) \ = \ \sum_{i,\bar{i},f,\bar{f}} {V_{\bar{\Delta}}}^{\bar{f}}{}_{\bar{i}}(\bar{z})\ {\bar{V}_\Delta}{}^{f}{}_{i}(z)\ \alpha_{i\bar{i}}{}^{f\bar{f}} \qquad . \tag{17}$$

For *all* fields in the diagonal (A-series) models Δ and $\bar{\Delta}$ coincide, and $\alpha_{i\bar{i}}{}^{f\bar{f}} = \delta_{i\bar{i}}\, \delta^{f\bar{f}}$, while for the D_{odd} models with $k = 2 \bmod 4$, $\bar{\Delta} = \sigma(\Delta)$, $\alpha_{i\bar{i}}{}^{f\bar{f}} = \delta_{i\sigma(\bar{i})}\, \delta^{f\sigma(\bar{f})}$, where $\sigma(I) = I$ for I integer and $\sigma(I) = k/2 - I$ for I half-odd integer.

The n-point functions in front of a crosscap can be obtained from the $2n$-point chiral conformal blocks, since the crosscap interchanges left and right chiral vertex operators with the same labels. More precisely, one can account for the crosscap by inserting in the amplitudes a "crosscap operator" that acts according to

$$\hat{C} \;=\; \sum_l \Gamma_l\; |\Delta_l >< \bar{\Delta}_l| \qquad , \tag{18}$$

where, as in ref. [8], Γ_l is the normalization of the one-point function of a primary field of conformal dimensions $(\Delta_l, \bar{\Delta}_l)$ in front of the crosscap. The coefficients Γ_l are a crucial ingredient of the construction, since their squares determine the vacuum Klein-bottle amplitude. The n-point functions in front of a crosscap are then defined as

$$< \phi_{\Delta_1,\bar{\Delta}_1} \,..\, \phi_{\Delta_n,\bar{\Delta}_n} >_C \;=\; < \hat{C}\phi_{\Delta_1,\bar{\Delta}_1} \,..\, \phi_{\Delta_n,\bar{\Delta}_n} >_0 \quad , \tag{19}$$

and can be expressed in terms of the chiral conformal blocks for $2n$ chiral vertex operators. Indeed, using eqs. (16) and (18),

$$\begin{aligned} < \phi_{\Delta_1,\bar{\Delta}_1}..\phi_{\Delta_n,\bar{\Delta}_n} >_C \;&=\; \sum_l \Gamma_l < 0|V_{\Delta_1}(z_1)..V_{\Delta_n}(z_n)|\Delta_l >< \bar{\Delta}_l|\bar{V}_{\bar{\Delta}_1}(\bar{z}_1)..\bar{V}_{\bar{\Delta}_n}(\bar{z}_n)|0 > \\ &=\; \sum_l \Gamma_l < 0|V_{\Delta_1}(z_1)..V_{\Delta_n}(z_n)\; V_{\bar{\Delta}_1}(\bar{z}_1)..V_{\bar{\Delta}_n}(\bar{z}_n)|0 >_l \quad . \end{aligned} \tag{20}$$

The (anti)symmetrization properties of the fields $\phi_{\Delta_i,\bar{\Delta}_i}$ with respect to the crosscap then imply the relation

$$< \phi_{\Delta_i,\bar{\Delta}_i}(z_i,\bar{z}_i)\; X >_C \;=\; \varepsilon_{(i,\bar{i})} < \phi_{\bar{\Delta}_i,\Delta_i}(\bar{z}_i, z_i)\; X >_C \quad , \tag{21}$$

where X is any polynomial in the fields. It should be noticed that ε coincides with the sign in the Klein-bottle projection (3) only for the integer-isospin fields, while it is opposite for the half-odd-integer ones, as befits the pseudoreality of the half-odd-integer isospin representations of $SU(2)$. Eq. (21) generally determines all the coefficients Γ_l. In particular, for all fields with $\varepsilon = -1$, as well as for those of non-zero spin (*i.e.* with

$\Delta \neq \bar{\Delta}$), Γ must vanish, since

$$
\begin{aligned}
< \phi_{\Delta,\bar{\Delta}}(z,\bar{z}) >_C \quad &= \quad \sum_l \Gamma_l < 0|V_\Delta(z)|\Delta_l >< \bar{\Delta}_l|\bar{V}_{\bar{\Delta}}(\bar{z})|0 > \\
&= \quad \Gamma_\Delta \; \delta_{\Delta,\bar{\Delta}} \;\; < 0|V_\Delta(z) \; V_{\bar{\Delta}}(\bar{z})|0 > = \; < \phi_{\bar{\Delta},\Delta}(\bar{z},z) >_C \quad . \qquad (22)
\end{aligned}
$$

Much more detailed information can be obtained by analyzing the two-point functions in front of a crosscap

$$
\begin{aligned}
&< \phi_{\Delta_1,\bar{\Delta}_1}(z_1,\bar{z}_1) \; \phi_{\Delta_2,\bar{\Delta}_2}(z_2,\bar{z}_2) >_C \\
&\qquad = \quad \sum_l \Gamma_l < 0|V_{\Delta_1}(z_1) \; V_{\Delta_2}(z_2)|\Delta_l >< \bar{\Delta}_l|\bar{V}_{\bar{\Delta}_1}(\bar{z}_1) \; \bar{V}_{\bar{\Delta}_2}(\bar{z}_2)|0 > \\
&\qquad = \quad \sum_l \Gamma_l < 0|V_{\Delta_1}(z_1) \; V_{\Delta_2}(z_2) \; V_{\bar{\Delta}_1}(\bar{z}_1) \; V_{\bar{\Delta}_2}(\bar{z}_2)|0 >_l \\
&\qquad = \quad \sum_l \Gamma_l \; C_{(\Delta_1,\bar{\Delta}_1)(\Delta_2,\bar{\Delta}_2)}{}^{(\Delta_l,\Delta_l)} \; S_l(z_1,z_2,\bar{z}_1,\bar{z}_2) \quad , \qquad (23)
\end{aligned}
$$

where S_l denote the normalized s-channel conformal blocks with a field of dimensions (Δ_l,Δ_l) in the intermediate channel, while $C_{(\Delta_1,\bar{\Delta}_1)(\Delta_2,\bar{\Delta}_2)}{}^{(\Delta_l,\Delta_l)}$ are the two-dimensional structure constants. Note that we have set $\bar{\Delta}_l = \Delta_l$, since only spin-zero fields are allowed as intermediate states and, for brevity, we have omitted the additional labels of the chiral vertex operators. In a similar fashion, one obtains

$$
\begin{aligned}
&< \phi_{\bar{\Delta}_1,\Delta_1}(\bar{z}_1,z_1) \; \phi_{\Delta_2,\bar{\Delta}_2}(z_2,\bar{z}_2) >_C \\
&\qquad = \quad \sum_l \Gamma_l < 0|V_{\bar{\Delta}_1}(\bar{z}_1) \; V_{\Delta_2}(z_2)|\Delta_l >< \bar{\Delta}_l|\bar{V}_{\Delta_1}(z_1) \; \bar{V}_{\bar{\Delta}_2}(\bar{z}_2)|0 > \\
&\qquad = \quad \sum_l \Gamma_l < 0|V_{\bar{\Delta}_1}(\bar{z}_1) \; V_{\Delta_2}(z_2) \; V_{\Delta_1}(z_1) \; V_{\bar{\Delta}_2}(\bar{z}_2)|0 > \\
&\qquad = \quad \sum_l \Gamma_l \; C_{(\bar{\Delta}_1,\Delta_1)(\Delta_2,\bar{\Delta}_2)}{}^{(\Delta_l,\Delta_l)} \; S_l(\bar{z}_1,z_2,z_1,\bar{z}_2) \quad . \qquad (24)
\end{aligned}
$$

The s-channel blocks in eqs. (23) and (24) can be related with the help of the duality matrices, as shown in figure 1.

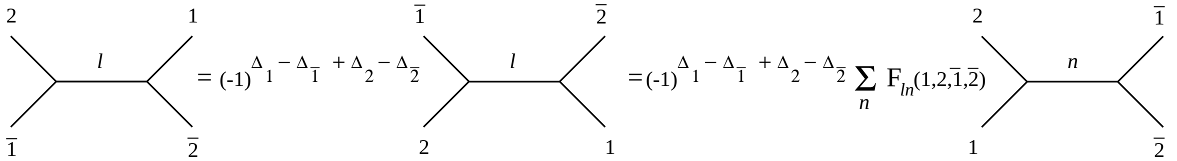

Figure 1

The first equality in the figure holds since the corresponding transformation is $B_1(B_3)^{-1}$, where B_i are the exchange matrices for the conformal blocks [11], while the second one follows from the definition of the fusion matrix F. Therefore

$$S_l(\bar{z}_1, z_2, z_1, \bar{z}_2) \;=\; (-1)^{\Delta_1-\bar{\Delta}_1+\Delta_2-\bar{\Delta}_2} \sum_n F_{ln}(1,2,\bar{1},\bar{2})\; S_n(z_1, z_2, \bar{z}_1, \bar{z}_2) \quad , \tag{25}$$

and inserting this expression in eqs. (23) and (24) and using eq. (21) yields a set of linear relations between the crosscap coefficients Γ_l,

$$\begin{aligned} \varepsilon_{(1,\bar{1})}\; (-1)^{\Delta_1-\bar{\Delta}_1+\Delta_2-\bar{\Delta}_2}\; \Gamma_n\; C_{(\Delta_1,\bar{\Delta}_1)(\Delta_2,\bar{\Delta}_2)}{}^{(\Delta_n,\Delta_n)} \;=& \\ \sum_l \Gamma_l\; C_{(\bar{\Delta}_1,\Delta_1)(\Delta_2,\bar{\Delta}_2)}{}^{(\Delta_l,\Delta_l)} F_{ln}(1,2,\bar{1},\bar{2}) \quad .& \end{aligned} \tag{26}$$

This "crosscap constraint" was first derived in ref. [8] where, however, only the $\varepsilon = 1$ case was considered.

The crosscap constraint can be looked at from two different viewpoints. On the one hand, if the structure constants C are known, eq. (26) can be used to determine the values of the one-point functions Γ_l. On the other hand, if the one-point functions are (even partially) known, for instance from the vacuum Klein-bottle amplitude, eq. (26) can be used to determine the structure constants in the model. In particular, it allows one to determine rather simply the relative signs of the structure constants in the diagonal and non-diagonal models.

Since the F matrix depends only on the chiral content of the theory, it is the same for the diagonal A models and for the non-diagonal D_{odd} models of the same level k. If, without any loss of generality, one orders the isospins of the fields in eq. (26) so that they satisfy

$$I_4 = \min\{I_i\} \ , \quad |I_{12}| \leq I_{34} \ , \quad |I_{23}| \leq I_{14} \ , \quad \text{where} \quad I_{ij} = I_i - I_j \qquad , \tag{27}$$

F has the form [12]

$$\begin{aligned} F_{I_{14}+r, I_{34}+s}(1,2,3,4) = \sum_{j=0}^{\min(s,m-r)} (-1)^{m-r-j}\; \frac{[2I_{34}+s+j]![I_{34}-I_{12}+s]!}{[2I_{34}+2s]![I_{34}-I_{12}+j]!} \times& \\ \frac{[s]![m-j]![m-j-I_{32}+I_{14}]![2I_{14}+2r+1]!}{[s-j]![j]![m-r-j]![r]![r-I_{32}+I_{14}]![2I_{14}+m+r-j+1]!} \quad ,& \end{aligned} \tag{28}$$

where $m = 2I_4$ and

$$
\begin{aligned}
[n] &= \frac{q^n - q^{-n}}{q - q^{-1}} \quad , \qquad\qquad q = exp(\frac{i\pi}{k+2}) \quad , \\
[n]! &= [n][n-1]! \quad , \qquad\qquad [0]! = [1]! = 1 \quad .
\end{aligned} \tag{29}
$$

Moreover,

$$
C_{(1,3)(2,4)}{}^{(I,I)} = \epsilon_I \sqrt{\frac{C_{12I} C_{34I}}{C_{II0}}} \quad , \tag{30}
$$

$$
C_{12I} = \frac{[I_1 + I_2 + I + 1]![I_1 + I_2 - I]![I_1 + I - I_2]![I_2 + I - I_1]!}{[2I_1]![2I_2]![2I]!} \quad , \tag{31}
$$

and $\epsilon_I = +1$ for the diagonal (A) models, while for the non-diagonal (D_{odd}) models

$$
\begin{aligned}
\epsilon_I &= 1 \qquad \text{for } \ \mathrm{I}_1, ..., \mathrm{I}_4 \ \ \text{integers} \\
\epsilon_I &= (-1)^I \qquad \text{for } \ \mathrm{I}_1, ..., \mathrm{I}_4 \ \ \mathrm{half-odd-integers} \quad .
\end{aligned} \tag{32}
$$

This exhausts all the cases we need, since in the D_{odd} models the one-point functions Γ vanish for all fields of half-odd-integer isospin. Hence I in eq. (32) is always an integer, while I_1 and I_3 are either both integer or both half-odd-integer, and the same is true for I_2 and I_4. It should be noticed that both F and the structure constants depend on normalization choices that, however, do not affect the crosscap constraint. We use a somewhat non-standard convention, so that the two-dimensional two-point functions on the sphere are normalized to the quantum dimensions of the fields, and therefore $C_{II0} = [2I + 1]$. This is a natural choice when one exhibits the internal quantum group symmetry of the model. A determination of the signs ϵ_I for the minimal models may be found in [13] and references therein.

The crosscap constraint is a vastly overdetermined system, and therefore here we shall only exhibit a set of equations sufficient to determine all the non-vanishing Γ_l. Let us therefore concentrate on the case $k = 4\rho + 2$, relevant to the non-diagonal $D_{2\rho+3}$ models. From the amplitudes for two fields of isospin $I_1 = \bar{I}_1 = I$, $I_2 = \bar{I}_2 = 1$, $(I = 1, .., k/2 - 1)$, present both in the A and in the D_{odd} models, one finds

$$
\Gamma_{I+1} \sqrt{[2I+3]} = \Gamma_I \frac{[2]}{[2I]} \sqrt{[2I+1]} \; + \; \frac{\Gamma_{I-1}}{\sqrt{[2I-1]}} \left([2I+1] - \frac{[2]}{[2I]} \right) \quad . \tag{33}
$$

On the other hand, from the amplitude for two fields of isospins $I = \bar{I} = k/4$ (hence half-odd-integer), one finds

$$\sum_{l=0}^{k/2} (-1)^l \, \Gamma_l \, \sqrt{[2l+1]} \; + \; \varepsilon_{(\frac{k}{4},\frac{k}{4})} \, \Gamma_0 \; [k/2+1] \; = \; 0 \tag{34}$$

for the A model, and

$$\sum_{l=0}^{k/2} \Gamma_l \, \sqrt{[2l+1]} \; + \; \varepsilon_{(\frac{k}{4},\frac{k}{4})} \, \Gamma_0 \; [k/2+1] \; = \; 0 \tag{35}$$

for the D_{odd} model, where ε is the same phase that enters eq. (26). These equations allow one to determine recursively all the coefficients Γ_l up to a common normalization factor that can be fixed by comparison with the Klein-bottle amplitude. The final result can be presented in a rather compact form in terms of the modular matrices S and P. For the real $D_{2\rho+3}$ model with $\varepsilon_{(I,I)} = 1$ for all fields,

$$\Gamma_a = \frac{(-1)^{\frac{a^2-1}{8}} \; P_{\frac{k}{2},a}}{\sqrt{S_{1,a}}} \quad , \tag{36}$$

while for the complex $D_{2\rho+3}$ model with $\varepsilon_{(I,I)} = -1$ for all half-odd-integer isospin fields,

$$\Gamma_a = \frac{(-1)^{\frac{a^2-1}{8}} \; P_{\frac{k}{2}+2,a}}{\sqrt{S_{1,a}}} \quad , \tag{37}$$

where, as in the previous Section, the weight a is related to the isospin I of the characters by $a = 2I + 1$. Since both $P_{\frac{k}{2},a}$ and $P_{\frac{k}{2}+2,a}$ vanish for even a [1], the phase factors in eqs. (36) and (37) are just signs. One can verify that these solutions satisfy all of eqs. (26).

Descendants of the D_{odd} Models

In this Section we construct the open descendants of the D_{odd} models. The starting point for the $D_{2\rho+3}$ model (with level $k = 4\rho + 2$) is the torus partition function [5]

$$T \; = \; \sum_{odd \;\, a=1}^{k+1} |\chi_a|^2 \; + \; \sum_{even \;\, a=2}^{k/2-1} (\chi_a \bar{\chi}_{k+2-a} + h.c.) \; + \; |\chi_{k/2+1}|^2 \quad . \tag{38}$$

The two sets of solutions of the crosscap constraint obtained in the previous Section imply that there are two different vacuum Klein-bottle amplitudes. Anticipating the real

(complex) structure of the Chan-Paton charge spaces for these models, we shall denote the two solutions corresponding to eqs. (36) and (37) by $\tilde{K}^r$ and $\tilde{K}^c$ respectively. These determine the two Klein-bottle projections,

$$K^r = \frac{1}{2}\left(\sum_{odd\ a=1}^{k+1} \chi_a - \chi_{k/2+1}\right) \quad \text{and} \quad K^c = \frac{1}{2}\left(\sum_{odd\ a=1}^{k+1} \chi_a + \chi_{k/2+1}\right) \quad . \tag{39}$$

As already stressed, the correspondence between the signs in K and the phases in eq. (21) is $\varepsilon_{(I,I)} = (-1)^{2I} K_{2I+1}$ where, as in eq. (3), K_a is the coefficient of χ_a in the Klein-bottle projection. In addition, for the fields that do not enter the Klein-bottle projection, ε is determined by the sign in the corresponding diagonal model. As for the D_5 model, one can determine the annulus and Möbius partition functions by turning on one charge at a time. The resulting maximal dimensions of the Chan-Paton charge spaces are equal to $\rho + 3$ both for the real and for the complex model with $k = 4\rho + 2$. This should be contrasted with the corresponding dimension for the diagonal models [1], equal to $k + 1$. In addition, the complex model contains only one pair of complex charges. Remarkably, as in the diagonal models, both the annulus and the Möbius partition functions may be expressed in terms of $k+1$ (now linearly dependent) charges, of the fusion-rule coefficients

$$N_{ab}{}^{c} = \sum_d \frac{S_{ad} S_{bd} S^{\dagger}{}_{cd}}{S_{1d}} \quad , \tag{40}$$

and of the integer-valued tensor Y

$$Y_{ab}{}^{c} = \sum_d \frac{S_{ad} P_{bd} P^{\dagger}{}_{cd}}{S_{1d}} \quad . \tag{41}$$

More precisely, for the real model

$$A^r = \frac{1}{2} \sum_{a,b,c} \chi_a\ N_{bc}{}^{k+2-a}\ n_b\ n_c \quad , \tag{42}$$

$$\tilde{A}^r = \frac{1}{2} \sum_a \chi_a\ (-1)^{a-1} \left(\frac{\sum_b\ S_{ab}\ n_b}{\sqrt{S_{1a}}}\right)^2 \quad , \tag{43}$$

$$M^r = \pm\frac{1}{2} \sum_{a,b} \hat{\chi}_a\ Y_{b\frac{k}{2}}{}^{a}\ n_b \quad , \tag{44}$$

$$\tilde{M}^r = \pm\frac{1}{2} \sum_{a,b} \hat{\chi}_a \left(\frac{P_{\frac{k}{2}a}\ S_{ab}}{S_{1a}}\right) n_b \quad . \tag{45}$$

In proving eqs. (42) and (44), one needs the relation

$$N_{bc}{}^{k+2-a} = \sum_d (-1)^{d-1} \frac{S^{\dagger}{}_{ad}\ S_{bd}\ S_{cd}}{S_{1d}} \quad , \tag{46}$$

implied by

$$\sum_b \; S_{ab}(-1)^{b-1} S_{bc} \;=\; \delta_{a+c,k+2} \quad . \tag{47}$$

The $k+1$ $(= 4\rho+3)$ charges n may then be expressed in terms of $\rho+3$ independent charges l as

$$n_a \;=\; n'_a \;+\; \frac{i}{2\sqrt{\rho+1}}\; O_a (-1)^{\frac{a-1}{2}} \; (l_{\rho+2} - l_{\rho+3}) \quad , \tag{48}$$

where O_a denotes the projector on odd a. In addition, the n' satisfy the relations

$$\begin{aligned} & n'_{\frac{k+2}{2}+a} \;=\; n'_{\frac{k+2}{2}-a} \quad , \qquad n'_{\frac{k+2}{4}+a} \;=\; -\, n'_{\frac{k+2}{4}-a} \qquad (a \geq 1) \quad , \\ & n'_a \;=\; -\,\frac{l_{a+1}}{2} \quad , \qquad (1 \leq a \leq \rho) \quad , \\ & n'_{\frac{k+2}{2}} \;=\; l_1 \quad , \qquad n'_{\frac{k+2}{4}} \;=\; \frac{1}{2}\,(l_{\rho+2} + l_{\rho+3}) \quad . \end{aligned} \tag{49}$$

The open sector of the complex model is described by

$$A^c \;=\; \frac{1}{2} \sum_{a,b,c} \chi_a \; {N_{bc}}^a \; n_b \; n_c \tag{50}$$

$$\tilde{A}^c \;=\; \frac{1}{2} \sum_a \chi_a \; \left(\frac{\sum_b \; S_{ab} \; n_b}{\sqrt{S_{1a}}} \right)^2 \tag{51}$$

$$M^c \;=\; \pm\, \frac{1}{2} \sum_{a,b} \hat{\chi}_a \; {Y_{b,\frac{k}{2}+2}}^a \; n_b \tag{52}$$

$$\tilde{M}^c \;=\; \pm\, \frac{1}{2} \sum_{a,b} \hat{\chi}_a \; \left(\frac{P_{\frac{k}{2}+2,a} \; S_{ab}}{S_{1a}} \right) n_b \quad , \tag{53}$$

with the same charge reduction of eq. (49) and, as usual, with the values of the two complex charges identified according to $l_{\rho+3} = \bar{l}_{\rho+2}$. Remarkably, all the coefficients of the l charges in the partition functions of eqs. (42), (44), (50) and (52) are again integer and satisfy the general consistency condition for the Möbius projection given in eq. (8). This presentation of the partition functions has the virtue of making the consistency with the fusion algebra manifest, whereas in the presentation in terms of independent charge sectors this crucial property of the construction is somewhat obscured. These expressions deserve two additional remarks. First of all, the involution

$${N_{bc}}^a \leftrightarrow {N_{bc}}^{k+2-a} \qquad \text{and} \qquad {Y_{bc}}^a \leftrightarrow {Y_{b,k+2-c}}^a \qquad , \tag{54}$$

interchanges the partition functions of the real and the complex model, and a similar relation holds for the crosscap coefficients Γ,

$$(\Gamma^r{}_a)^2 \;=\; (\Gamma^c{}_{k+2-a})^2 \qquad . \tag{55}$$

Moreover, when expressed in terms of the charges n, the annulus partition functions of all real(complex) off-diagonal models coincide with the partition functions of the corresponding complex(real) diagonal ones. The origin of both these correspondences is still to be better understood.

Descendants of the E_7 Model

We now proceed to describe the open and unoriented sectors of the E_7 model. It has level $k = 16$ and torus partition function

$$T_{E_7} \;=\; |\chi_1|^2 + |\chi_3|^2 + |\chi_4|^2 + |\chi_5|^2 + (\chi_6\bar{\chi}_2 + h.c.) \quad , \tag{56}$$

in terms of the generalized characters corresponding to the extended symmetry of the D_{10} model, whose partition function in this notation reads

$$T_{D_{10}} \;=\; \sum_{i=1}^{6} |\chi_i|^2 \qquad . \tag{57}$$

The Klein-bottle projection of the E_7 model is

$$K_{E_7} \;=\; \frac{1}{2}\,(\chi_1 + \chi_3 + \chi_4 + \chi_5) \qquad , \tag{58}$$

and the coefficients in the corresponding vacuum-channel amplitude $\tilde{K}$ cannot be expressed as ratios of single matrix elements of P and S, as was the case for the D_{odd} models. Rather, the squares of the crosscap coefficients are linear combinations of such terms. The signs of these coefficients are again determined by the crosscap constraint of eq. (26). As in the D_{odd} models, both the annulus and the Möbius partition functions of the descendants of the E_7 model

$$A_{E_7} \quad = \quad \frac{1}{2}\,\Big(\chi_1(l_1^2 + l_2^2 + l_3^2 + l_4^2) \;+$$

$$
\begin{aligned}
&(\chi_2+\chi_6)(l_3^2+l_4^2+2l_1l_4+2l_2l_3+2l_2l_4+2l_3l_4) \ + \\
&\chi_3(l_2^2+l_3^2+2l_4^2+2l_1l_2+2l_1l_3+2l_2l_3+2l_2l_4+4l_3l_4) \ + \\
&\chi_4(l_2^2+2l_3^2+2l_4^2+2l_1l_3+2l_1l_4+2l_2l_3+4l_2l_4+4l_3l_4) \ + \\
&\chi_5(l_1^2+l_2^2+l_3^2+2l_4^2+2l_1l_2+2l_2l_3+2l_3l_4)\Big)
\end{aligned} \tag{59}
$$

$$
\begin{aligned}
M_{E_7} \quad = \quad \pm\,\frac{1}{2}\Big(&\hat{\chi}_1(l_1+l_2+l_3+l_4) \ + \ (\hat{\chi}_6-\hat{\chi}_2)(l_3+l_4) \ + \\
&\hat{\chi}_3(l_2+l_3+2l_4) \ + \ \hat{\chi}_4\,(-l_2) \ + \ \hat{\chi}_5(l_1+l_2+l_3)\Big)
\end{aligned} \tag{60}
$$

can be obtained by charge reduction from the corresponding expressions for the D_{10} model. Indeed, if in the annulus amplitude

$$
A_{D_{10}} \ = \ \frac{1}{2}\sum_{a,b,c}\chi_a \ N_{bc}{}^a n_b n_c \tag{61}
$$

the six charges n are expressed in terms of the four independent charges l according to

$$
\begin{aligned}
n_1 \ &= \ \frac{1}{3\sqrt{3}}\ (-2l_1+4l_2-l_4) \quad , \quad n_2 \ = \ n_6 \ = \ \frac{1}{3\sqrt{3}}\ (-l_1-l_2+3l_3+l_4) \quad , \\
n_3 \ &= \ \frac{1}{3\sqrt{3}}\ (4l_1+l_2+2l_4) \quad , \quad n_4 \ = \ \frac{1}{3\sqrt{3}}\ (-l_1+2l_2+4l_4) \quad , \\
n_5 \ &= \ \frac{1}{3\sqrt{3}}\ (2l_1+2l_2+3l_3-2l_4) \quad ,
\end{aligned} \tag{62}
$$

one recovers the E_7 amplitude of eq. (59). One may also recover the Möbius amplitude of eq. (60) provided its transverse channel is defined in terms of the proper Γ_a.

We have thus completed the classification of the open descendants for all the ADE $SU(2)$ modular invariants of Cappelli, Itzykson and Zuber. In particular, we have shown that all models with non-diagonal left-right pairing can be obtained by charge reduction from the corresponding diagonal ones. One can envisage further extensions, in particular to $SU(3)$, cosets and supersymmetric models, with many possible applications to open-string theories.

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